CHAPTER 3

Measures of Goodness of a Set of Colour Scanning Filters

This chapter addresses the problem of determining the accuracy of colour measurements obtained from a set of filters. The literature review in Chapter 2 indicates that a measure of a set of colour scanning filters could be used to define an optimality criterion for the design of a 'good' set of filters. The requirements of a measure are presented in this chapter and a data-independent measure that satisfies these requirements is introduced. The basis of this measure is the relation between the HVISS and the vector space spanned by the scanning filters. The measure is related to the principal angles between the space defined by the filters and the HVISS.

The definition of a measure is very closely linked to the means of judging the quality of a colour reproduction. Commonly used error measures for a colour reproduction include the mean-square CIE tristimulus error and the average ΔE_{Lab} error in CIE $L^*a^*b^*$ space (see section 1.1.2). The error can also be measured as the euclidean distance in a space that is a linear transformation of CIE tristimulus space, i.e. in a tristimulus space defined by tristimulus values which are not necessarily CIE tristimulus values. Such an error could also be represented as a weighted euclidean norm in CIE tristimulus space. While all spaces which are non-singular linear transformations of CIE tristimulus space are three-dimensional, and hence isomorphic as vector spaces, the euclidean distance between two points in the spaces represents different measures of the difference between the corresponding colours. For example, if the scanner measurements are to be used for a display on a colour CRT, the CIE tristimulus values are not the required measurements (see section 2.3). The required measurements are, instead, a linear transformation of the CIE tristimulus values. The space where a colour is represented by the three tristimulus values with respect to the primaries of the CRT is a linear transformation of CIE tristimulus space. The error vector is hence a linear transformation of the CIE tristimulus error vector, and the euclidean norm of the error vector in CIE tristimulus space is not necessarily identical to its euclidean norm in the other space. The HVSS, for example, consists of points whose coordinates may be defined by the tristimulus values with respect to a set of orthogonal functions that are a linear transformation of the CIE matching functions (for example, MacAdam's orthogonal matching functions, [20]). CIE tristimulus space and the HVSS are hence linear transformations of each other, and distances between two points are not preserved by the transformation.

The generalization of the mean-square CIE tristimulus error to the mean-square value of the error vector in any linear transformation of the HVISS results in a family of data-dependent measures. The data-independent measure is shown to be related to the family of data-dependent measures. The measures are shown to be generalizations of the quality factor of Neugebauer [22]. Simulations are presented to demonstrate the effectiveness of the measures and to compare their performance. These simulations also indicate the correlation of the respective measures with the perceptual error measure ΔE_{Lab}, which is the 2-norm of the error vector in CIE $L^*a^*b^*$ space.

The chapter is organised as follows. The requirements of a measure of goodness are defined in section 1. The analysis of an error measure leads to a data-independent measure of goodness in section 2. It is possible that a set of three scanning filters that spans the HVISS is not realizable due to some practical limitations of the fabrication process. In such a situation, it is necessary to look for a set of four or more scanning filters that does span the HVISS. A perfect set of four scanning filters is presented in section 3. It is demonstrated that the q-factor of a single filter is not a good indicator of the appropriateness of the filter as a member of a set of more than three colour scanning filters. Experimental results comparing actual filter-set performance with the data-independent measure of goodness are presented in section 4. The experiments are simulated using colour ensembles from a lithographic printer, a thermal printer, an inkjet printer, a colour copier and a set of Munsell chips. The results demonstrate the effectiveness of the introduced measure. The measure is seen to be a good indicator of average mean-square error and average error in $L^*a^*b^*$ space for the signal ensembles used. Data-dependent measures are introduced in section 5. Simulation results are presented to compare the performance of the formulated data-dependent measures with the data-independent measure. A summary of the major results of the chapter is presented in section 6.

3.1 Requirements of an Effective Measure

The problem of colour scanning may be formulated as the problem of determining the CIE tristimulus values, or the 3-vector $\mathbf{A}_L^T\mathbf{f}$ (see section 1.1.1). Similarly, the problem of colour correction may be formulated as the problem of determining the 3K values $[\mathbf{A}_{L_1}, \mathbf{A}_{L_2}, ...\mathbf{A}_{L_K}]^T\mathbf{f}$. Consider, for example, the problem of determining the 6-stimulus vector consisting of the CIE tristimulus values with respect to the two illuminants, $\mathbf{L}_1$ and $\mathbf{L}_2$. This problem has applications in colour correction [31, 38]. The 6-stimulus vector consists of the CIE tristimulus values under two viewing illuminants, for example a daylight illuminant like D65 and a fluorescent illuminant like

F2. Another problem with applications in colour reproduction is that of determining the projection of a reflective spectrum onto a P-dimensional space, the P defining parameters being the most important parameters for colour correction with respect to a number of different viewing illuminants [39]. The problem of determining the spectral signatures of portions of the earth's surface may be expressed as the problem of determining $\mathbf{S}^T\mathbf{f}$, where the columns of $\mathbf{S}$ represent the responses of sensors used for remote sensing (see section 1.3).

The problems of the design of filters for colour scanning, colour correction and remote sensing may hence be defined in a common framework. The common problem is the design of filters to obtain the values $\mathbf{V}^T\mathbf{f}$ where $\mathbf{f}$ is an N-vector representing the signal to be measured (visual stimulus for the problems of colour scanning and correction, or radiation received by sensors for the remote sensing application). The matrix $\mathbf{V}$ consists of s columns. These columns represent the combined effect of the CIE matching functions and a viewing illuminant in the case of colour scanning, or the combined effect of the CIE matching functions and many different viewing illuminants in the case of colour correction. In the application to remote sensing, the columns of $\mathbf{V}$ represent the sensor responses. In all cases, the designed filters do not need to replicate the columns of $\mathbf{V}$, and it is sufficient to obtain measurements from which the values $\mathbf{V}^T\mathbf{f}$ may be determined through a linear transformation.

The problem of colour scanning may hence be generalized to the problem of obtaining the set of s values

$$\mathbf{t} = \mathbf{V}^T\mathbf{f} \tag{3.1}$$

In the above expression, $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_s]$, and $\mathbf{f}$ is an N-vector. The vector $\mathbf{t}$ may be referred to as the s-stimulus vector.

When a colour reproduction is to be viewed under illuminant $\mathbf{l}$, the CIE tristimulus values for illuminant $\mathbf{l}$ determine the visual stimulus of the reproduced signal completely [3]. This implies that if the signal $\mathbf{g}$ is a reproduction of $\mathbf{f}$ such that

$$\mathbf{A}_L^T\mathbf{f} = \mathbf{A}_L^T\mathbf{g}$$

i.e. if $\mathbf{g}$ is a metamer of $\mathbf{f}$ under illuminant $\mathbf{l}$, the human eye will not detect a difference in the colours of $\mathbf{g}$ and $\mathbf{f}$ under viewing illuminant $\mathbf{l}$. In the case when the display is a colour CRT, the reproduction $\mathbf{g}$ is produced by a linear combination of the (usually three) primaries of the colour display (see section 2.3). If $\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, ...\mathbf{p}_s]$ represents the s primaries then (see section 2.3)

$$\mathbf{g} = \mathbf{P}\mathbf{p}$$

and the problem is to determine the (usually tristimulus) values $\mathbf{p}$ such that $\mathbf{g}$ is a metamer of $\mathbf{f}$. This implies that for accurate reproduction under a particular illuminant, the goal is not always the determination of the CIE tristimulus values under that illuminant. For example, the goal could be the determination of the RGB values of a colour stimulus (see section 1.1.1). It has been shown in section 2.3, equation (2.6) that $\mathbf{p}$ is a linear transformation of $\mathbf{A}_L^T\mathbf{f}$. In general, the goal is the determination of a linear transformation of the CIE tristimulus values [31, 28] (see section 2.7.8).

Let $\{\mathbf{v}_i\}_{i=1}^s$ denote a set of vectors (not necessarily linearly independent) that define the space to be spanned. This space is denoted by $R(\mathbf{V})$, where $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2....\mathbf{v}_s]$. It may be noted that the information obtained through the s-stimulus vector in equation (3.1) is equivalent to knowing the projection of $\mathbf{f}$ onto $R(\mathbf{V})$ [28]. The goal of the scanning problem may be restated as the goal of obtaining the projection of $\mathbf{f}$

onto $R(\mathbf{V})$ to take into account the different tristimulus values that are required for different applications as mentioned above. As all tristimulus values with respect to a particular illuminant are within a linear transformation of each other, and of the projection onto the HVISS, this serves to standardize the measurement procedure. The projection onto $R(\mathbf{V})$ is referred to as the fundamental, analogous to the fundamental of a colour signal in the specific instance of $\mathbf{V} = \mathbf{A}_L$ [31, 3], (see section 2.7.8). If $\mathbf{X}^-$ represents a generalized inverse of $\mathbf{X}$, a mathematical expression for the fundamental is [26]:

$$P_V\mathbf{f} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^-\mathbf{V}^T\mathbf{f} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^-\mathbf{t} \qquad (3.2)$$

which is clearly a linear transformation of the s-stimulus values. Equation (2.2) provides an expression for the fundamental when $R(\mathbf{V})$ is the HVISS.

It may be noted that the linear transformation mentioned is non-invertible (unless the spanning set $\{\mathbf{v}_i\}_{i=1}^s$ is of rank N, which is not usual in any of the applications discussed earlier). In cases where $\mathbf{V}$ is not of full rank, i.e. the set $\{\mathbf{v}_i\}_{i=1}^s$ is not linearly independent, the generalized inverse of $\mathbf{V}^T\mathbf{V}$ is not unique. The projection operator, however, is unique [26, pg. 110]. In the specific case when the set $\{\mathbf{v}_i\}_{i=1}^s$ is linearly independent, the generalized inverse of $\mathbf{V}^T\mathbf{V}$ is unique and is the multiplicative inverse of $\mathbf{V}^T\mathbf{V}$. In this more familiar case, the expression for the fundamental is $P_V\mathbf{f} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\mathbf{f} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{t}$. It may also be noted that the projection operator does not preserve distances.

If a set of effective scanning filters (see section 2.7.8 for a definition of 'effective scanning filters') is linearly independent, the measurements obtained, $\{\mathbf{Hm}_i)^T\mathbf{f}\}_{i=1}^r$ will be non-redundant and can be used to obtain an estimate of the s-stimulus values. If these filters are not linearly independent, all measurements are not necessary to find the estimate if the measurements are noise-free and hence consistent. The extra measurements may help eliminate noise in the scanning process. In either case, the s-stimulus values may be completely determined iff the space spanned by the effective scanning filters includes the space $R(\mathbf{V})$. This may be shown in a manner analogous to the way equation (2.14) was derived in section 2.7.8. It is also shown in the Appendix, Theorem 2.

An effective measure of goodness of a set of scanning filters should satisfy the following conditions:

1. A measure of goodness of the filter set is related to the space spanned by the scanning filters and the relation of this space with the HVISS. The measure should depend only on the space spanned by the scanning filters and not on particular, individual filters.

2. The measure should not be related to the noise performance of the filters, as it should measure performance of the filters with respect to an error that occurs independently of any additional measurement noise.

3. For perfect scanning, $R(\mathbf{V})$ should be contained in the space spanned by the scanning filters. Hence the measure should indicate a perfect set of scanning filters. When the scanning is not perfect, the measure should distinguish among filter sets according to the goodness of the approximation to projections onto $R(\mathbf{V})$.

4. The measure should be generalizable to an arbitrary number of filters and an arbitrary reproduction space.

It is necessary to note here that $R(\mathbf{V})$ need not be three-dimensional; for example, effective colour correction requires the projection onto a space of dimension greater than three as mentioned earlier in this section [38, 39, 31]. The dimensions of $R(\mathbf{V})$ and $R(\mathbf{M}_H)$ may not be equal; a large set of filters $\{\mathbf{m}_i\}_{i=1}^{r}$ may be used to ensure the spanning of $R(\mathbf{V})$ as mentioned in section 2.7.8 and as demonstrated in section 3.3.

A measure that immediately comes to mind is the dimension of the intersection of $R(\mathbf{M}_H)$ and $R(\mathbf{V})$. On closer examination this measure is found to be too coarse. It takes on only integer values, and does not distinguish well enough between 'good' and 'not-so-good' sets of scanning filters. This problem is illustrated in Fig. 3.1 for a hypothetical three-dimensional spectral space with a two-dimensional HVISS. The spaces spanned by two different sets of scanning filters are shown. It can be seen that the projection onto $R(\mathbf{M}_2)$, denoted $P_{M2}\mathbf{f}$, is a better approximation to the fundamental, denoted $P_V\mathbf{f}$, than is the projection onto $R(\mathbf{M}_1)$, denoted $P_{M1}\mathbf{f}$. This is not indicated by the dimension of the intersection of the spaces with the HVISS, in both cases this dimension is one. Hence it can be seen that the dimension of interesection is not a measure of the 'angle' between the spaces.

For an example of this problem in an N-dimensional space, consider the space spanned by the N-vectors

$$\begin{aligned} \mathbf{v}_1 &= [1,0,0,0,......0]^T \\ \mathbf{v}_2 &= [0,1,0,0,......0]^T \\ \mathbf{v}_3 &= [0,0,1,0,......0]^T \end{aligned} \tag{3.3}$$

as the space to be spanned. Let the N-vectors

$$\mathbf{Hm}_1 = [1,0,0,0,......0]^T$$

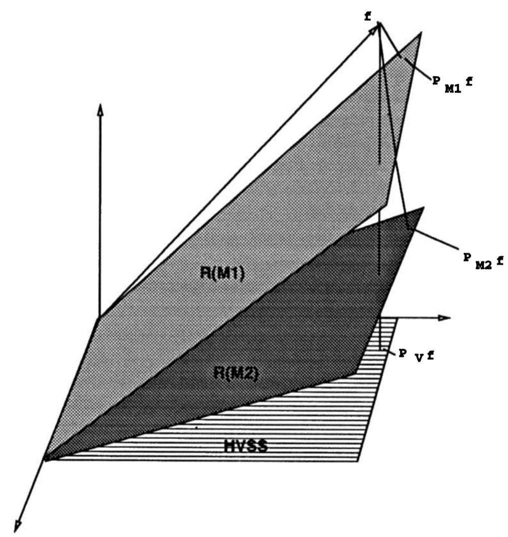

Figure 3.1: Dimension of intersection is not a valid measure

$$\begin{aligned} \mathbf{Hm}_2 &= [0,0,1,0,......0]^T \\ \mathbf{Hm}_3 &= [0,0,0,1,......0]^T \end{aligned} \tag{3.4}$$

be the effective scanning filters. The dimension of the intersection, $R(\mathbf{V}) \cap R(\mathbf{M}_H)$ is two. Now suppose

$$\mathbf{Hm}_1' = [1,0,0,0,......0]^T,$$

$$\mathbf{Hm}_2' = [0,0.95,0,0.05,........0]^T,$$

and

$$\mathbf{Hm}_3' = [0,0,1,0,......0]^T$$

is another set of scanning filters. The dimension of the intersection is also two, but the second set will provide much more information of the required projection (for example, information about the second coordinate of the signal can be obtained more reliably from the second set of scanning filters than from the first). A measure is needed that will distinguish between such sets of filters.

A measure of the goodness of a set of scanning filters is directly related to the error associated with such a set in the absence of any additional noise. The reproduction error is defined as the difference between the required signal and the signal obtained with the scanning filter set. This error depends on the particular reflectance vector, $\mathbf{f}$, which is being measured. As the human eye is sensitive only to errors in the Human Visual Subspace, the reproduction error may be defined as the difference between the actual and the reconstructed fundamentals.

Common measures of colour reproduction error are the square fundamental error, the square CIE tristimulus error and the error in CIE $L^*a^*b^*$ space. The square fundamental error is the square of the 2-norm of the reproduction error vector. As the CIE tristimulus vector is a linear transformation of the fundamental, the tristimulus error vector is a linear transformation of the fundamental error vector. The square CIE tristimulus error is the square of the 2-norm of the CIE tristimulus error vector. As the linear transformation does not preserve distances, the square CIE tristimulus error is generally different from the square fundamental error. The error in CIE $L^*a^*b^*$ space is the 2-norm of the error vector in CIE $L^*a^*b^*$ space. The transformation from tristimulus space to CIE $L^*a^*b^*$ space (see section 1.1.2) is non-linear. As a result, there is no well-defined transformation from the error vector in tristimulus space to the error vector in CIE $L^*a^*b^*$ space, and a particular tristimulus error vector results in a non-unique error vector in CIE $L^*a^*b^*$ space.

It is common to consider an average of the error measure over some well-defined set of reflectance spectra, $\{\mathbf{f}_k\}$. The averages of all three error measures are studied in this chapter. The mean-square error between fundamentals in N-space will now be related to a data-independent error measure which has the desired properties of a measure of a set of colour scanning filters.

3.2 An Error Measure and a Related Measure of Goodness

Two major error measures are often used in the evaluation of colour reproductions, mean-square fundamental error and mean $L^*a^*b^*$ error. In either case, the mean error indicates an average over a particular data set and is hence dependent on the data set. The problems of correlating mean-square error with perceptual error are well known. The mean-square error is addressed because it is easy to manipulate and because its analysis provides valuable insight into the problem of reproduction errors due to filter-construction errors. The mean ΔE_{Lab} error measure is far more difficult to analyse and manipulate, but is a valid perceptual error measure. While there are cases where colour estimates may have low mean-square errors and high ΔE_{Lab} errors and vice versa, the average of the errors over a data set are generally in qualitative agreement, as demonstrated in sections 3.4 and 3.5.5.

Before an error expression can be obtained, some notation needs to be established. Let an orthonormal basis for $R(\mathbf{V})$ be defined by $\mathbf{N} = [\mathbf{n}_1 \; \mathbf{n}_2 ....\mathbf{n}_\alpha]$ such that $R(\mathbf{N}) = R(\mathbf{V})$ and:

$$\mathbf{N}^T\mathbf{N} = \mathbf{I}. \tag{3.5}$$

Such a basis may be obtained by the Gram-Schmidt orthogonalisation procedure or the QR factorization [12]. The number of orthonormal vectors, α, is the rank of $\{\mathbf{v}_i\}_{i=1}^{s}$ and α equals s iff $\{\mathbf{v}_i\}_{i=1}^{s}$ is a linearly independent set. Similarly, define an orthonormal basis for $R(\mathbf{M}_H)$ by $\mathbf{O} = [\mathbf{o}_1 \; \mathbf{o}_2 ....\mathbf{o}_\beta]$ such that $R(\mathbf{O}) = R(\mathbf{M}_H)$

and:

$$\mathbf{O}^T\mathbf{O} = \mathbf{I}. \tag{3.6}$$

Again, notice that β is the rank of $\{\mathbf{m}_i\}_{i=1}^r$ and β equals r iff $\{\mathbf{m}_i\}_{i=1}^r$ is a linearly independent set. The orthonormal bases $\mathbf{N}$ and $\mathbf{O}$ need not represent realisable filters. Let $P_{M_H}(.)$ represent the orthogonal projection operator (with respect to the euclidean norm) onto $R(\mathbf{M}_H)$ in N-space. Similarly, let $P_V(.)$ represent the orthogonal projection operator (with respect to the euclidean norm) onto $R(\mathbf{V})$ in N-space. Define $P_V(\mathbf{O}) = [P_V(\mathbf{o}_1)\ P_V(\mathbf{o}_2).......P_V(\mathbf{o}_\beta)]$ and likewise $P_{M_H}(\mathbf{N}) = [P_{M_H}(\mathbf{n}_1)\ P_{M_H}(\mathbf{n}_2)...$ $....P_{M_H}(\mathbf{n}_\alpha)]$. The projection of a visual spectrum $\mathbf{f}$ onto $R(\mathbf{V})$ is the information required for accurate colour reproduction as defined by the designers of the reproduction system. From equations (3.2) and (3.5), this required projection of a visual spectrum $\mathbf{f}$ onto $R(\mathbf{V})$ is

$$P_V(\mathbf{f}) = \mathbf{N}\mathbf{N}^T\mathbf{f} \tag{3.7}$$

In the special case where $R(\mathbf{V})$ is the HVISS, $P_V(\mathbf{f})$ is the fundamental of the spectrum $\mathbf{f}$ as defined in [3].

A linear transformation $\mathbf{B}$ is often used to correct the scanner measurements for scanning filter errors and to compensate for the characteristics of a particular ensemble (see section 2.2). The corrected scanning filter measurements are $\hat{\mathbf{t}}_1 = \mathbf{B}\mathbf{M}_H^T\mathbf{f}$ where the matrix $\mathbf{B}$ is chosen such that $E[||\mathbf{V}^T\mathbf{f} - \mathbf{B}\mathbf{M}_H^T\mathbf{f}||^2]$ is minimum over a particular ensemble, or so that the scanner measurements 'best fit' the required values. The transformation $\mathbf{B}$ is clearly dependent on the data set. The data correction gives corrected scanning filter data

$$\hat{\mathbf{t}}_1 = (\mathbf{V}^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^-)(\mathbf{M}_H^T\mathbf{f}) \tag{3.8}$$

where $\mathbf{R} = E[\mathbf{f}\mathbf{f}^T]$ is the sample correlation matrix of the ensemble. The corrected scanning filter data is the best (minimum mean square error) linear estimate of the s-stimulus values of $\mathbf{f}$ (see Appendix, Theorem 1). The above expression reduces to expression (2.4) if the space to be spanned is the HVISS. If $\mathbf{R}$ is invertible and $\mathbf{M}_H$ is full-rank, then $(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^-$ is unique and $(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^- = (\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^{-1}$. In general, $(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^-$ is not unique. The estimate in equation (3.8) is unique [26, pg. 110]. It may be shown that if $R(\mathbf{M}_H) \supseteq R(\mathbf{V})$, i.e. if $\mathbf{V} = \mathbf{M}_H\mathbf{X}$ for some matrix $\mathbf{X}$, then $\hat{\mathbf{t}}_1 = \mathbf{V}^T\mathbf{f} = \mathbf{t}$, and the estimate is perfect (see Appendix, Theorem 2).

The fundamental may be estimated from the corrected data. This estimate will represent a signal which has the corrected data as its tristimulus values. As demonstrated in sections 3.4, 3.5.5 and [35] the correction reduces both mean-square and $L^*a^*b^*$ errors considerably. The estimated fundamental is

$$(\widehat{P_V\mathbf{f}})_1 = (\mathbf{V}(\mathbf{V}^T\mathbf{V})^-)\hat{\mathbf{t}}_1 = P_V(\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^-\mathbf{M}_H^T)\mathbf{f} \tag{3.9}$$

which gives

$$(\widehat{P_V\mathbf{f}})_1 = (\mathbf{V}(\mathbf{V}^T\mathbf{V})^-\mathbf{V}^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^-)(\mathbf{M}_H^T\mathbf{f}) \tag{3.10}$$

As the fundamental is a linear transformation of the s-stimulus vector, the above estimate of the fundamental is the linear minimum mean-square error estimate and the above expression for the fundamental reduces to expression (2.5) when the space to be spanned is the HVISS.

3.2.1 Mean Square Error

The difference between the fundamental $P_V(\mathbf{f})$ and its estimate from the corrected scanning filter measurements, $(\widehat{P_V\mathbf{f}})_1$, is the reproduction error. From equations (3.9) and (3.2) the reproduction error is:

$$\mathbf{e} = P_V\mathbf{f} - (\widehat{P_V\mathbf{f}})_1 = P_V(\mathbf{I} - \mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^{-1}\mathbf{M}_H^T)\mathbf{f} = \mathbf{V}(\mathbf{V}^T\mathbf{V})^{-}(\mathbf{t} - \hat{\mathbf{t}}_1) \tag{3.11}$$

It can be shown that (see Appendix, Theorem 2)

$$\mathbf{e} = \mathbf{0} \ \textit{for all } \mathbf{R} \quad <=> \quad R(\mathbf{M}) \supseteq R(\mathbf{V}) \tag{3.12}$$

and perfect reproduction for all possible data sets is possible if and only if the space spanned by the scanning filters includes the space to be spanned. The mean-square value of the error in equation (3.11) over the particular data set can be shown to be (see Appendix, Theorem 3)

$$E[||\mathbf{e}||^2] = Trace(\mathbf{N}^T\mathbf{R}\mathbf{N} - \mathbf{N}^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^{-}\mathbf{M}_H^T\mathbf{R}\mathbf{N}) \tag{3.13}$$

3.2.2 Bounds on the Error Expression

Bounds on the error expression (3.13) may be obtained in terms of the structure of the matrix $\mathbf{R}$. It can be shown that (see Appendix, Theorem 4)

$$0 \leq E[||\mathbf{e}||^2] \leq Trace\ P_V\mathbf{R} \leq Trace\ \mathbf{R}$$

To obtain some heuristic understanding of the meaning of these bounds, consider a simple example demonstrating the extreme cases of zero-error and of maximum error. Let the space to be spanned be defined by the set $\{\mathbf{v}_i\}_{i=1}^3$ defined in section 3.1, equation (3.3). Let the scanning filter set be the set $\{\mathbf{m}_i\}_{i=1}^3$ defined in section 3.1, equation (3.4). Notice that in this particular case, the effective scanning filter set is orthonormal. The set $\{\mathbf{v}_i\}_{i=1}^3$ is also orthonormal. This implies that $\mathbf{M}_H = \mathbf{O}$

and $\mathbf{N} = \mathbf{V}$. Let $\mathbf{R}$ be a diagonal matrix with only one non-zero diagonal element. Let this element be w^2. Then,

$$Trace\ \mathbf{R} = w^2$$

and from equation (3.13)

$$E[||e||^2] = R_{22}$$

where R_{22} is the second element on the diagonal of the matrix $\mathbf{R}$. The case of zero-error occurs when the non-zero element of the diagonal of $\mathbf{R}$ is any element other than the second. The maximum error occurs when the non-zero element of the diagonal of $\mathbf{R}$ is the second. For example, this happens when the ensemble of spectra for this synthetic example is defined by

$$\{\mathbf{f} \mid \mathbf{f}_i = u \ if \ i = 2, \ \mathbf{f}_i = 0 \ else\}$$

where u is a gaussian random variable with variance σ^2 and mean m. The value of the maximum error in this case is $\sigma^2 + m^2$.

From the above example it can be seen that the case of zero-error occurs when the energy in the signal that is in the space to be spanned is also in the space spanned by the scanning filters. In the special case when the space that needs to be spanned is the HVISS, the zero-error case will occur when all the energy in the signal that is in the HVISS is also in the space spanned by the scanning filters. Similarly, the maximum error occurs when all the energy of the signal 'escapes' the scanning filters, but lies in the space to be spanned.

From the above observations it is clear that the correlation structure of the signal determines the error to a large extent. When nothing is known about the correlation structure, it is common to assume that $\mathbf{R}$ is a scalar multiple of the identity matrix, and that the signal power is equally distributed in all directions. This assumption implies that assumed concentration of signal power in a particular direction will not bias the error to make it maximum or minimum as in the cases just discussed. The assumption leads to an error measure that is demonstrated to be particularly useful, and is independent of the data set.

3.2.3 An Error Measure

In the particular case when $\mathbf{R}$ is a scalar multiple of the identity matrix, i.e. when the spectrum $\mathbf{f}$ can be expressed as a sequence of independent, identically distributed, random variables, the error expression is considerably simplified. With this assumption, the error expression is independent of the data set and can be related to the q-factor measure [22]. This assumption is often made when no information is available about the signal statistics, and the random variables of the signal are assumed independent of each other. It is a 'maximum ignorance' assumption for signal estimation.

Substitution of $\mathbf{R} = \sigma^2\mathbf{I}$ into equation (3.8) gives the following estimate of the tristimulus values:

$$\hat{\mathbf{t}}_2 = (\mathbf{V}^T\mathbf{M}_H(\mathbf{M}_H^T\mathbf{M}_H)^-)(\mathbf{M}_H^T\mathbf{f}) = \mathbf{V}^T P_{M_H}\mathbf{f} \tag{3.14}$$

Similarly, from equation (3.10) the fundamental of $\mathbf{f}$ may be estimated as

$$(\widehat{P_V}\mathbf{f})_2 = \mathbf{V}(\mathbf{V}^T\mathbf{V})^-\mathbf{V}^T\mathbf{M}_H(\mathbf{M}_H^T\mathbf{M}_H)^-(\mathbf{M}_H^T\mathbf{f}) = \mathbf{V}(\mathbf{V}^T\mathbf{V})^-\hat{\mathbf{t}_2} = P_V P_{M_H}\mathbf{f} \tag{3.15}$$

Note that if $\mathbf{M}$ is of full-rank, $\mathbf{M}_H^T\mathbf{M}_H$ is invertible. From equations (3.2), (3.5) and (3.6) the above estimate of the fundamental is

$$\widehat{P_V(\mathbf{f})}_2 = \mathbf{N}\mathbf{N}^T\mathbf{O}\mathbf{O}^T\mathbf{f} \tag{3.16}$$

and is known as the 'uncorrected' estimate. It is the relevant information about $\mathbf{f}$ that can be obtained from the scanned data.

The term 'uncorrected' indicates that though the scanning filter data is altered to take into account the fact that the scanning filters may not replicate the $\mathbf{v}_i$. The data manipulation does not 'correct' the measurements for correlation weighting of the data. In the special case where $R(\mathbf{V})$ is the HVISS, the manipulation accounts for the fact that the space spanned by the scanning filters is an approximation to the HVISS. The manipulation, however, does not take into consideration the correlation of the data, as this is assumed unknown.

Expressions (3.7) and (3.16) give the required projection and the obtained projection, respectively. The error vector is the difference between the two expressions:

$$\mathbf{e} = \mathbf{N}\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{f}. \tag{3.17}$$

and is identical to expression (3.11) if $\mathbf{R} = \sigma^2\mathbf{I}$, as expected. Error expression (3.13) may be shown to be (Appendix, Theorem 5)

$$E[||\mathbf{e}||^2] = \sigma^2(Trace(P_V - P_V P_{M_H})) = \sigma^2\alpha(1 - \frac{Trace(P_V P_{M_H})}{\alpha}) \tag{3.18}$$

which is (Appendix, Theorem 5)

$$E[||\mathbf{e}||^2] = \sigma^2(\sum_{i=1}^{\alpha}(1 - \lambda_i^2(\mathbf{O}^T\mathbf{N}))) = \sigma^2\alpha(1 - \frac{\sum_{i=1}^{\alpha}\lambda_i^2(\mathbf{O}^T\mathbf{N})}{\alpha})$$

where σ^2 is the variance of a single component of $\mathbf{f}$ and $\lambda_i(\mathbf{O}^T\mathbf{N})$ denotes the i^{th} singular value of $\mathbf{O}^T\mathbf{N}$. It is appropriate to mention that

$$\lambda_i^2(\mathbf{O}^T\mathbf{N}) = Cos^2(\theta_i) \quad i = 1, \; .... \; \alpha$$

where θ_i is the i^{th} principal angle between $R(\mathbf{V})$ and $R(\mathbf{M}_H)$ [12]. Note that the values $1 - \lambda_i^2(\mathbf{O}^T\mathbf{N}), \quad i = 1, ... \alpha$ are the eigenvalues of $\mathbf{I}_\alpha - \mathbf{N}^T\mathbf{O}\mathbf{O}^T\mathbf{N}$ which may be rewritten as:

$$\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)(\mathbf{I} - \mathbf{O}\mathbf{O}^T)\mathbf{N} = \mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T)(\mathbf{N}^T(\mathbf{I} - \mathbf{O}\mathbf{O}^T))^T$$

from equations (3.5) and (3.6). As the above expression is a quadratic form,

$$1 - \lambda_i^2(\mathbf{O}^T\mathbf{N}) \geq 0 \quad i = 1, ... \alpha$$

and

$$0 \leq \lambda_i^2(\mathbf{O}^T\mathbf{N}) \leq 1 \quad i = 1, ... \alpha$$

which implies that

$$0 \leq E[||\mathbf{e}||^2] = \sigma^2\alpha(1 - \frac{\sum_{i=1}^{\alpha} \lambda_i^2(\mathbf{O}^T\mathbf{N})}{\alpha}) \leq \sigma^2\alpha. \tag{3.19}$$

Perfect reproduction is possible when:

$$\alpha = \sum_{i=1}^{\alpha} \lambda_i^2(\mathbf{O}^T\mathbf{N})$$

or,

$$\lambda_i(\mathbf{O}^T\mathbf{N}) = 1 \quad i = 1, ... \alpha. \tag{3.20}$$

It may be noted that the conditions for perfect reproduction are independent of the variance of the components of $\mathbf{f}$ and are identical to those in (3.12) (see Appendix, Theorem 5). The term α in expression (3.19) is the dimension of the space to be spanned and is assumed invariate.

The summation in expression (3.19) may be used as a measure of goodness of the filters, as a high value of the summation indicates a low error. A normalised measure of goodness of the filter set is:

$$\nu(\mathbf{V}, \mathbf{M}_H) = \frac{\sum_{i=1}^{\alpha} \lambda_i^2(\mathbf{O}^T\mathbf{N})}{\alpha} \tag{3.21}$$

As

$$\alpha = Trace\mathbf{N}^T\mathbf{N} = Trace\mathbf{N}\mathbf{N}^T = TraceP_V$$

the above definition may also be written as (from Appendix, Theorem 5)

$$\nu(\mathbf{V}, \mathbf{M}_H) = \frac{TraceP_V P_{M_H}}{TraceP_V} = \frac{TraceP_V P_{M_H}(\sigma^2\mathbf{I})}{TraceP_V(\sigma^2\mathbf{I})} \tag{3.22}$$

From equations (3.19) and (3.21) the mean-square error between the actual and reconstructed fundamentals is

$$E[||\mathbf{e}||^2] = \sigma^2\alpha(1 - \nu(\mathbf{V}, \mathbf{M}_H)) \tag{3.23}$$

if the components of the reflectance spectra are such that $E[\mathbf{ff}^T] = \sigma^2\mathbf{I}$. The error is zero if and only if the set of filters is perfect and equation (3.20) holds. Perfect reproduction implies and is implied by

$$\nu(\mathbf{V}, \mathbf{M}_H) = 1 \tag{3.24}$$

The error measure can be shown to be related to the principal angles between the two subspaces, $R(\mathbf{V})$ and $R(\mathbf{M}_H)$. Principal angles are well-known in Numerical Linear Algebra and this relationship is discussed in detail in section 3.2.5.

It can be shown that the measure ν satisfies the requirements stated in section 3.1:

1. The definition of the measure, expression (3.22), indicates that the dependence on P_{M_H} is the only dependence on the scanning filters in the expression for the

measure ν. The projection operator P_{M_H} depends only on $R(\mathbf{M}_H)$ and not on individual filters. The measure is hence related to the spaces spanned by the filters and the space to be spanned, and not on individual filters.

2. The measure ν is not related to the noise performance of the filters, as the error model for the measure is based on the error of equation (3.17) which does not include any measurement noise.

3. The measure is directly related to the error between the real and estimated fundamentals as indicated by equation (3.23). Equation (3.24) indicates that perfect reproduction is indicated by a value of unity of the measure.

4. As the measure is dependent only on P_V and P_{M_H}, and hence on $R(\mathbf{V})$ and $R(\mathbf{M}_H)$, it is generalizable to an arbitrary number of filters and an arbitrary reproduction space.

3.2.4 Relationship with q-factor

It can be shown that the measure of goodness defined in equation (3.21) is related to Neugebauer's q-factor measure [22], discussed in section 2.9. Specifically, this measure of goodness is the sum of the q-factors of the vectors $\mathbf{o}_i$ divided by the dimension of $R(\mathbf{V})$ (see Appendix, Theorem 6). Thus the sum of q-factors of the filters represented by $\mathbf{O}$, an orthonormal basis for the space spanned by the scanning filters, is a valid measure of goodness for the set of filters when the signal $\mathbf{f}$ is from an ensemble of independent, identically distributed random variables. The sum may be normalised so that a maximum value of unity indicates perfect reproduction. This normalised form is:

$$\nu(\mathbf{V}, \mathbf{M}_H) = \frac{\sum_{i=1}^{\beta} q(\mathbf{o}_i)}{\alpha} = \frac{\sum_{i=1}^{\beta} q(\mathbf{o}_i)}{Trace P_V} \tag{3.25}$$

Here the term q-factor is used in a general sense to mean the norm of the projection of the normalised vectors $\mathbf{o}_i$ onto the α -dimensional $R(\mathbf{V})$. Neugebauer's original definition of the q-factor implies the specific case where this space is the three-dimensional HVISS [22]. From equation (2.16), this is also

$$\nu(\mathbf{V}, \mathbf{M}_H) = \frac{\sum_{i=1}^{\beta} \sum_{j=1}^{\alpha} \mathbf{t}_{ij}^2}{\sum_{i=1}^{N} \sum_{j=1}^{\alpha} \mathbf{u}_{ij}^2} \tag{3.26}$$

where $\mathbf{t}_{ij}$ is the α-stimulus value of $\mathbf{o}_i$ with respect to $\mathbf{n}_j$ (the j^{th} vector in an orthogonal basis for $R(\mathbf{V})$), and $\mathbf{u}_{ij}$ is the α-stimulus value of $\mathbf{e}_i$ (the i^{th} column of $\mathbf{I}$ or the i^{th} vector in an orthogonal basis for N-space) with respect to $\mathbf{n}_j$.

A necessary condition for the measure to be independent of the particular set of filters used to span the respective spaces is that the filters $\mathbf{O}$ should be orthonormal. So, for example, $\frac{\sum_{i=1}^{r} q(\mathbf{Hm}_i)}{r}$ cannot be used as a measure instead of $\frac{\sum_{i=1}^{\beta} q(\mathbf{o}_i)}{\alpha}$. This is because filters with high values of $|(\mathbf{Hm}_i)^T\mathbf{Hm}_j|$ for $i \neq j$ may have high q-factors but poor joint performance. Ensuring that the filters $\mathbf{O}$ are orthogonal removes the effect of the 'correlation' ($|(\mathbf{Hm}_i)^T\mathbf{Hm}_j|$ for $i \neq j$). To see that the normalised sum of the q-factors of the effective scanning filters defined in equation (3.27) is not a valid measure, consider the following example (refer to Figs. 3.2-3.3 for an illustration).

Let $\mathbf{v}_1 = [1,0,0]^T$ and $\mathbf{v}_2 = [0,1,0]^T$ be the vectors representing a simple two-dimensional HVISS. Let the scanner characteristic defined in section 2.7.8 be uniform so that $\mathbf{HM} = \mathbf{M}$. Let $\mathbf{m}_1 = [1,0,0]^T$ and $\mathbf{m}_2 = [0, Cos(\pi/3), Sin(\pi/3)]^T$ be vectors representing the scanning filters. The filters are orthonormal and the normalised sum of the q-factors is .75. Now, suppose another set of scanning filters is chosen that spans the same space as the first set. Suppose the set is $\mathbf{m}_1 = [1, 0, 0]$ and $\mathbf{m}_2 = [\frac{2}{\sqrt{5}}, \frac{Cos(\pi/3)}{\sqrt{5}}, \frac{Sin(\pi/3)}{\sqrt{5}}]$. Notice that this set of scanning filters is not orthogonal. The normalised sum of q-factors for this set is .925 which is artificially high.

Let

$$\psi(\mathbf{V}, \mathbf{M}_H) = \frac{\sum_{i=1}^{r} q(\mathbf{H}\mathbf{m}_i)}{r} \tag{3.27}$$

Thus $\psi(\mathbf{V}, \mathbf{M}_H)$ defined in equation (3.27) is seen to be dependent on the particular filters used and is not just a function of the space spanned by the filters, unless the orthogonality condition is imposed on the filters. Hence it is not a valid measure of goodness. Note that if the effective filters $\mathbf{H}\mathbf{m}_i$ are narrowband with little overlap, they are 'close' to orthogonal. In this case $\psi(\mathbf{V}, \mathbf{M}_H)$ may be a good approximation to $\nu(\mathbf{V}, \mathbf{M}_H)$. The measure $\nu(\mathbf{V}, \mathbf{M}_H)$ avoids this problem entirely. Another example where the average q-factor does not indicate filter performance accurately may be found in section 3.4. Engledrum [7] has used the sum of q-factors to define a measure of goodness for a set of colour scanning filters. As mentioned in section 2.9, the average q-factor is inadequate in part because it cannot be extended to a set of more than three filters, and in part because it does not evaluate a set of three filters correctly. The reasons for the latter have just been discussed.

Note that for $\alpha = 1$ the proposed measure ν reduces to Neugebauer's q-factor. If each scanning filter is to be evaluated by itself, and not as part of a larger set of filters, the proposed measure can be used to evaluate the set consisting of a single scanning filter. In this case, the proposed measure is exactly the same as the q-factor of Neugebauer, and Neugebauer's definition of the q-factor may be seen as a specific instance of the measure defined in (3.21).

3.2.5 Relation to Principal Angles

As mentioned in section 3.2.3, the idea of the principal angles between two subspaces is well-known in Numerical Linear Algebra. It is important to see the connection between the measure developed in this chapter and this well-established mathematical concept. The principal angles $\theta_1,\ \theta_2,\ ....\ \theta_\alpha\ \ \epsilon\ [0, \pi/2]$ between $R(\mathbf{M}_H)$ and $R(\mathbf{V})$ are defined recursively by [12]:

$$cos(\theta_k) \;=\; \max_{\mathbf{u}\ \epsilon\ R(\mathbf{V})}\ \max_{\mathbf{v}\ \epsilon\ R(\mathbf{M}_H)}\ \mathbf{u}^T\mathbf{v} \;=\; \mathbf{u}_k^T\mathbf{v}_k$$

subject to:

$$||\mathbf{u}|| \;=\; ||\mathbf{v}|| \;=\; 1$$

$$\mathbf{u}^T\mathbf{u}_i \;=\; 0 \quad i \;=\; 1, ...k \;-\; 1$$

$$\mathbf{v}^T\mathbf{v}_i \;=\; 0 \quad i \;=\; 1, ...k \;-\; 1$$

It can be shown that [12]:

$$cos(\theta_k) \;=\; \lambda_k(\mathbf{O}^T\mathbf{N})$$

From equation (3.21) this implies that

$$\nu(\mathbf{V}, \mathbf{M}_H) \;=\; \frac{\sum_{i=1}^{\alpha} cos^2(\theta_i)}{\alpha}$$

Notice that the measure is large when the angles between the subspaces are small. It is interesting that a measure developed using the mean-square error is consistent with the observation that the error in the scanning process is larger if the angle between the subspaces is larger.

3.2.6 Relation to $L^*a^*b^*$ error

The mean-square error is not directly related to the $L^*a^*b^*$ error, and it is possible that reconstructions with low mean-square errors have high $L^*a^*b^*$ errors and vice versa. However, the average $L^*a^*b^*$ error over an ensemble of signals used in colorimetric experiments is usually monotonic as a function of the mean-square error. Heuristically this is reasonable. The $L^*a^*b^*$ transformation is based on the tristimulus values; so small errors in the tristimulus values will usually imply small $L^*a^*b^*$ errors. Experience with the proposed measure has indicated that small differences do not always yield errors in the same order as the measure would indicate, larger differences always have. The results of the experiments with several different ensembles demonstrate this when using data correction.

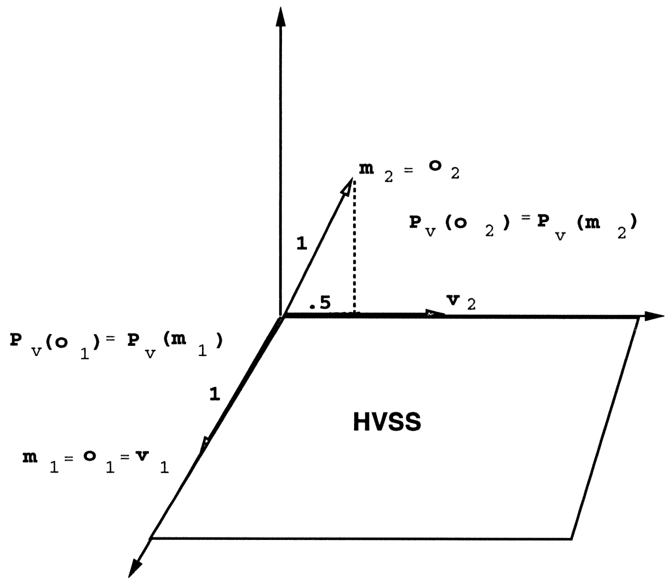

Figure 3.2: Measure as sum of q-factors of orthogonal filters

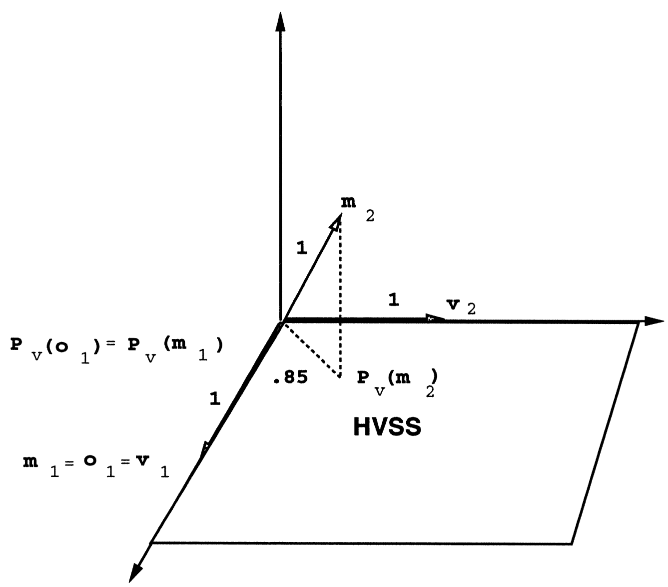

Figure 3.3: Measure is not sum of q-factors of non-orthogonal filters

3.3 A Perfect Filter Set

It was mentioned in section 3.1 that it is possible that a four-filter set can ensure perfect colour scanning when no combination of three filters span the HVISS. An example is presented here of an imperfect three-filter set with fairly high individual q-factors. It is demonstrated that the addition of a fourth filter to the set makes the set perfect, though the q-factor of the fourth filter is less than 0.25.

Consider the set of three all-positive filters shown in Fig. 3.4-3.6. This set of filters has a measure, $\nu(\mathbf{A}, \mathbf{M}_H)$, of 0.953. The filters have q-factors of 0.873, 0.891 and 0.939 respectively. As the measure is less than unity, this set does not span the HVSS, and is not a perfect set of scanning filters. The mean square error between the fundamental and the estimated uncorrected fundamental for the Munsell chip set is 0.188, and the average ΔE_{Lab} error for the same set is 3.4. The addition of a fourth filter to the set, shown in Fig. 3.7, increases the measure to 1.0, and the resulting four-filter set spans the HVSS. The q-factor of the fourth filter is 0.246. This is an example of a case where the q-factor of a filter is not indicative of its appropriateness as a scanning filter.

Consider the problem of selecting or designing scanning filters from a specified set of filters, for example, the Kodak Wratten set of gelatin filters. If only three filters are to be used in an attempt to span the HVISS, then all three must have high q-factors. This means that filter candidates can be limited. If more than three filters are to be used, then individual q-factors are useless (this is demonstrated in the experimental results of section 3.4) and the possible number of filters that must be considered is limited only by the total number in the set to be used for construction. The optimal design of such filter sets is an open problem. Another open problem is the design of

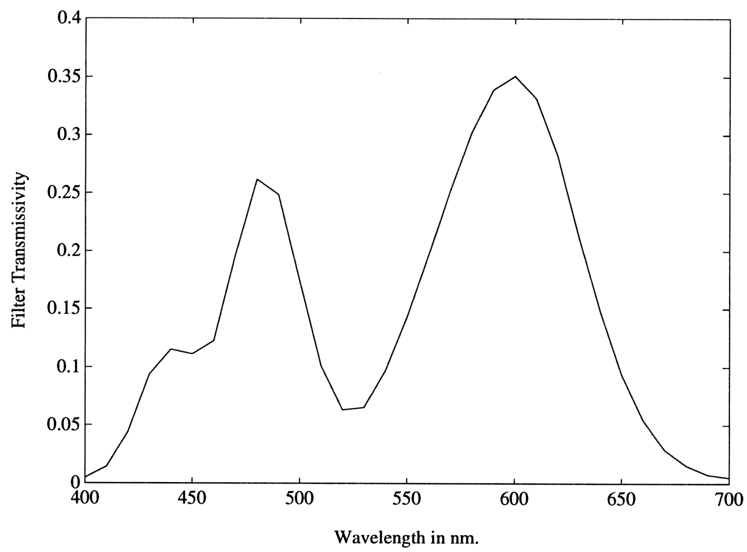

Figure 3.4: Filter 1-q-factor:0.873

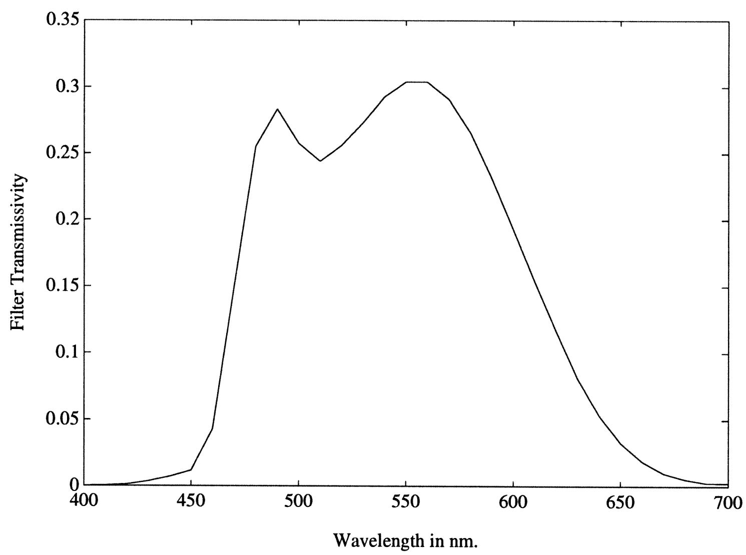

Figure 3.5: Filter 2-q-factor:0.891

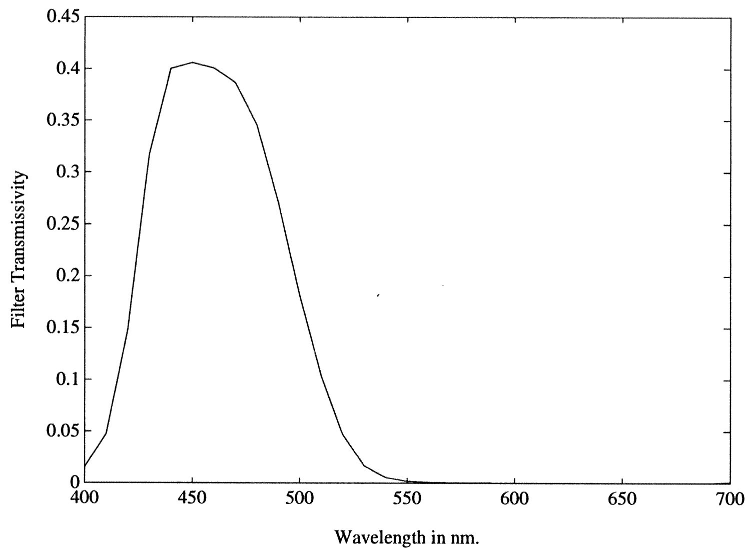

Figure 3.6: Filter 3-q-factor:0.939

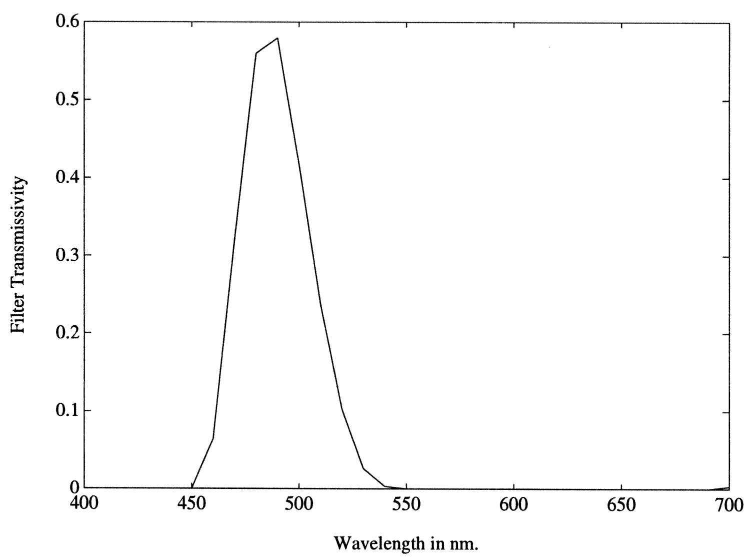

Figure 3.7: Filter 4-q-factor:0.246

filter sets using cascaded filters or sums of filters to form a single filter, as discussed in section 2.6.

3.4 Experimental Results

Several ensembles were used to study the appropriateness of the proposed measure, $\nu(\mathbf{V}, \mathbf{M}_H)$. The mean-square error and the ΔE_{Lab} error were calculated on sets of signals from a lithographic printer, a printer using a thermal dye-transfer process, an inkjet printer, a colour copier and a set of Munsell chips. The first four ensembles are representative of the range of printed materials that would be scanned in a publishing or copying application. The Munsell chip set is the only one not generated by a three or four colour process but by a more diverse set of pigments. The viewing and the recording light sources were assumed uniform, i.e. $\mathbf{L} = \mathbf{I}$, $\mathbf{H} = \mathbf{I}$, $\mathbf{M}_H = \mathbf{M}$ and $\mathbf{A}_L = \mathbf{A}$.

The filter set that is shown here is representative of several that were used in experiments during this work. Filter Set 1 consists of Kodak Wratten gelatin filter Nos. 52, 49, 2E. Filter Set 2 consists of Kodak Wratten gelatin filter Nos. 52, 49, 13. Filter Set 3 is the set of Kodak Wratten gelatin filter Nos. 52, 49, 72B. Filter Set 4 is the above set with a fourth filter added, the Kodak Wratten gelatin filter No. 57. Filter Set 5 consists of five filters, the four in Set 4 and Kodak Wratten gelatin filter No. 30. Table 3.1 lists the q factors of the individual filters.

In the first experiment, the fundamental was reconstructed directly from the uncorrected scanning filter data. Hence the estimated fundamental was calculated from equation (3.15). The mean-square error between this reconstructed fundamental and the true fundamental, $P_V\mathbf{f}$, is represented by e_1 in column 5 in Tables 3.2-3.6. The

ΔE_{Lab} error calculated on the basis of tristimulus values estimated from $P_M\mathbf{f}$, i.e. from equation (3.14), is represented by E_1 in column 6 in Tables 3.2-3.6. The white point is chosen as the point closest to being the 'brightest' (having the highest tristimulus values) and 'white' (with tristimulus values being approximately equal). The maximum ΔE_{Lab} error is represented by E_{1max} in column 7 in Tables 3.2-3.6. The data manipulations in the first experiment are independent of the particular data set.

In the second experiment, the scanning filter data was corrected to best fit the actual tristimulus values, assuming knowledge of second-order signal statistics. The ΔE_{Lab} error using (3.8) as the estimated tristimulus values is represented by E_2 in column 9 in Tables 3.2-3.6. The mean-square error between the reconstructed fundamental in (3.10) and the actual fundamental is represented by e_2 in column 8 in Tables 3.2-3.6. The maximum ΔE_{Lab} error is represented by E_{2max} in column 10 in Tables 3.2-3.6. The mean-square errors e_1 and e_2, the ΔE_{Lab} errors E_1 and E_2, the maximum ΔE_{Lab} errors E_{1max} and E_{2max}, the average q-factors, the sum of the q-factors, and the measure $\nu(\mathbf{A}, \mathbf{M})$ are tabulated in Tables 3.2-3.6 for the five sets of signals used.

Table 3.1: q-factors

Kodak Wratten Filter No.	q factor
52	0.9930
49	0.9563
2E	0.7252
13	0.6875
72B	0.6273
57	0.9032
30	0.5679

Table 3.2: Colour Copier Data Set. White Point = [7.84, 7.85, 7.79]

Filter Set	Measure ν	Sum of q-factors	Average q-factor	e_1	E_1	E_{1max}	e_2	E_2	E_{2max}
1	0.771	2.6746	0.892	0.016	10.76	34.88	0.00741	6.56	27.02
2	0.818	2.6369	0.879	0.022	11.61	25.04	0.00402	4.67	20.56
3	0.858	2.5766	0.859	0.050	12.96	26.84	0.00121	2.94	13.48
4	0.913	3.4799	0.870	0.054	14.62	26.16	0.00005	0.49	2.00
5	0.943	4.0478	0.810	0.002	2.72	4.91	0.00002	0.41	1.83

Table 3.3: Lithographic Printer Data Set. White Point = [7.32, 7.38, 6.99]

Filter Set	Measure ν	Sum of q-factors	Average q-factor	e_1	E_1	E_{1max}	e_2	E_2	E_{2max}
1	0.771	2.6746	0.892	0.011	7.74	23.69	0.00365	3.94	20.97
2	0.818	2.6369	0.879	0.018	8.94	21.76	0.00220	3.03	17.08
3	0.858	2.5766	0.859	0.070	14.32	26.46	0.00125	2.56	12.60
4	0.913	3.4799	0.870	0.073	15.38	25.57	0.00007	0.54	2.33
5	0.943	4.0478	0.810	0.002	2.40	4.41	0.00001	0.31	1.53

Table 3.4: Thermal Printer Data Set. White Point = [8.99, 8.98, 9.13]

Filter Set	Measure ν	Sum of q-factors	Average q-factor	e_1	E_1	E_{1max}	e_2	E_2	E_{2max}
1	0.771	2.6746	0.892	0.017	9.50	40.65	0.00981	7.49	59.42
2	0.818	2.6369	0.879	0.020	7.05	33.86	0.00709	6.35	58.63
3	0.858	2.5766	0.859	0.061	12.58	36.58	0.00086	2.65	12.70
4	0.913	3.4799	0.870	0.049	11.74	28.36	0.00066	1.81	7.13
5	0.943	4.0478	0.810	0.003	2.68	7.59	0.00007	0.87	5.50

Table 3.5: Inkjet Printer Data Set. White Point = [8.18, 8.23, 7.93]

Filter Set	Measure ν	Sum of q-factors	Average q-factor	e_1	E_1	E_{1max}	e_2	E_2	E_{2max}
1	0.771	2.6746	0.892	0.065	15.15	41.76	0.0263	9.29	36.13
2	0.818	2.6369	0.879	0.101	18.57	36.78	0.0207	8.45	31.96
3	0.858	2.5766	0.859	0.096	15.89	32.83	0.0012	2.30	7.26
4	0.913	3.4799	0.870	0.097	16.48	31.88	0.0005	1.21	4.30
5	0.943	4.0478	0.810	0.011	6.84	15.07	0.0002	0.70	2.04

Table 3.6: Munsell Chip Set. White Point = [10.2025, 10.1897, 10.0349]

Filter Set	Measure ν	Sum of q-factors	Average q-factor	e_1	E_1	E_{1max}	e_2	E_2	E_{2max}
1	0.771	2.6746	0.892	0.044	8.55	25.91	0.0162	5.12	18.64
2	0.818	2.6369	0.879	0.035	6.57	22.97	0.0121	4.32	15.11
3	0.858	2.5766	0.859	0.188	16.41	31.14	0.0022	2.10	8.63
4	0.913	3.4799	0.870	0.152	15.73	29.31	0.0011	1.03	8.80
5	0.943	4.0478	0.810	0.004	2.43	7.43	0.0002	0.49	2.07

Notice that the mean-square error and the ΔE_{Lab} error exhibit the same trends in both experiments (i.e. e_1 and E_1 behave similarly, and so do e_2 and E_2). A conclusion is that the average mean-square error is a reasonable indicator of the average perceptual error for the ensembles studied here. It should be noted here that there could be individual points where the square error between fundamentals is low, and the ΔE_{Lab} error is high, or vice versa, but that the average values are follow the same trend.

The proposed measure predicts an improvement in average performance when the number of filters is increased, as in the increase of filters from Set 3 to Set 4 to Set 5. This improvement is also indicated by the sum of the q-factors of the individual filters, although it is not indicated by the average q-factor (for example, Filter Set 5 has a lower average q-factor but a higher measure and better performance as indicated by both mean-square and ΔE_{Lab} errors). The sum of q-factors will increase monotonically as the number of filters increases, and so will the performance of a filter set, if data correction is employed. There does not seem to be any other relationship between the sum of the q-factors and filter set performance.

As the proposed measure is an accurate theoretical indicator of mean-square error when the signal is composed of independent random variables, it is possible that it may not be as accurate an indicator of mean-square error when the signal is correlated. In fact, this may be observed in the uncorrected experimental results for Filter Sets 1, 2, and 3. Here, the average errors do not monotonically decrease as the measure increases. From the measurements with Filter Set 4 and the copier data set, the lithographic printer data set and the inkjet printer data set, it is seen that an increase in the number of filters in these cases increases the average error. This is because the data set is correlated and the estimates used are not the best estimates for the particular data set. Hence, an increase in the number of filters does not ensure that the additional information is used appropriately for the particular data set.

For a simple example illustrating the above observation, consider a simple one-dimensional HVISS in a three-dimensional space. This example is illustrated in Figure 3.8. Let the HVISS be the space spanned by the vector $[1, 0, 0,]^T$. Let the signal to be scanned be $\mathbf{f} = [1, 2, 1]^T$. The fundamental of this signal is $[1, 0, 0]^T$. Let the scanning filter set consist of the single filter $[\sqrt{0.5}, \sqrt{0.5}, 0]^T$. The estimated fundamental for a single scanning filter and without data correction is $[1.5, 0, 0]^T$. Suppose a second scanning filter is added to the set. Let this filter be $[\sqrt{0.5}, 0, \sqrt{0.5}]^T$. The estimated

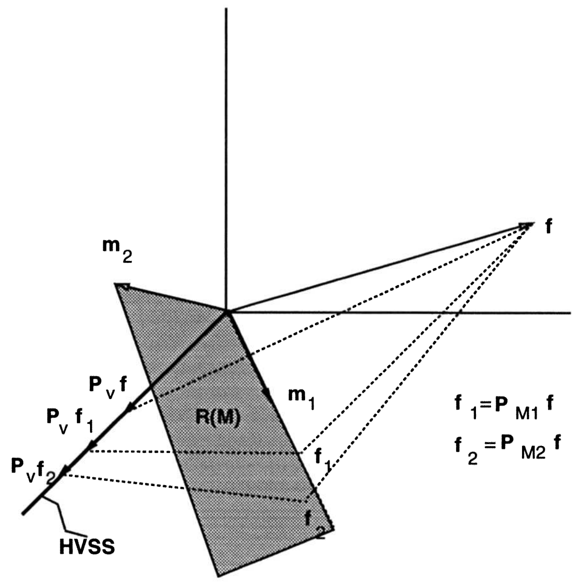

Figure 3.8: Increase in Number of Filters does not Provide a Better Estimate

fundamental without data correction is $[1.67, 0, 0]^T$. It is seen that the second estimate is further from the true fundamental than is the first estimate. Whether the projection of a data-point onto one space ($R(\mathbf{M}_H)$, for example) is closer to the projection onto another space ($R(\mathbf{V})$, for example) is highly data-dependent. Hence it is not necessarily true that a larger space $R(\mathbf{M}_H)$ ensures 'better' results for a particular data set when the data set is uncorrected, and this is illustrated by the simple example above. While this effect is 'cancelled out' on the average if the data set consists of signals with iid components, it accumulates if the data set does not consist of signals with iid components.

In contrast, it is seen that the goodness measure is a reasonable indicator of the error when the data set is corrected. The correlation of the measure and the error (both mean-square and perceptual) is remarkable for the three-filter sets: Filter Sets

1, 2 and 3. The average q-factor is shown to be a poor indicator of performance with respect to both mean-square and perceptual error in these cases.

The measure correlates well with the behaviour of the maximum ΔE_{Lab} error for corrected measurements, except for the experimental results using Filter Set 4 and the Munsell Chip data set. The measure is not meant to indicate the behaviour of the maximum error because it is based on mean-square error. The observations indicate that the measure should not be used for "fine tuning" a set of filters but can give a good indication of performance for larger differences in measure when knowledge of the signal statistics is not available.

3.5 Data-Dependent Measures

The value of the data-independent measure lies in the fact that it characterizes a filter set independent of a particular application, and that it does so by measuring the 'closeness' of the two spaces, $R(\mathbf{V})$ and $R(\mathbf{M}_H)$. When the data set is known and fixed, a measure that predicts the performance of the set of scanning filters on that particular data set is potentially more useful. This measure would be based on the error expression (3.11), the error after data correction. Such a measure would not treat each direction in N-space equally, but would 'weight' the directions depending on the correlation of the data. This section deals with the error when the statistics of the data set are fixed and known. A measure is developed in much the same manner in which the data-independent measure in section 3.2 and [35] was developed. The measure is data-dependent as it is derived from the error due to data-dependent manipulation of measurements.

Two measures of goodness that take into account the data set are presented.

One measure is based on the mean-square error between actual and reconstructed fundamentals and the other on mean-square error between tristimulus values. The error between fundamentals and the error between tristimulus values are within a linear transformation of each other. Because of this, the data-dependent measures are generalized to error measures which are based on the mean-square value of the euclidean norm of the error vector in any linear transformation of the HVISS. For example, tristimulus space is a linear transformation of the HVISS. It is shown that the ΔE_{Lab} error may not be simplified easily to provide a measure based on perceptual error. Simulations comparing the data-dependent measures with the data-independent measure are presented.

3.5.1 Data-Dependent Measure Based on Error Between Fundamentals

The equation for the uncorrected fundamental, equation (3.15), was used to obtain the expression for the data-independent measure of equation (3.21). A similar data-dependent measure would be based on the fundamental estimate of equation (3.9), which is obtained from the corrected filter measurements. For equations (3.15) and (3.9) to be written in a similar form, the distance measure for equation (3.15) should take into account the correlation of the data. In the case of data correction, the distance measure does not weight each direction equally, as does the distance measure for the case without data correction. To see this, define the inner-product

$$< \mathbf{x}, \mathbf{y} >' = \mathbf{x}^T \mathbf{R}^{-1} \mathbf{y} \tag{3.28}$$

Note that $\mathbf{R}$ is assumed invertible. Consider the norm induced by this inner product

$$||\mathbf{x}||'^2 = < \mathbf{x}, \mathbf{x} >' = \mathbf{x}^T \mathbf{R}^{-1} \mathbf{x}$$

Let P'_X be the orthogonal projection operator with respect to this norm. It can be shown that (see Appendix, Theorem 7)

$$\widehat{P_V \mathbf{f}_2} = P_V P'_{RM_H} \mathbf{f} \tag{3.29}$$

and that equation (3.13) is (using $trace\mathbf{XYZ} = trace\mathbf{ZXY}$)

$$E[||\mathbf{e}||^2] = Trace(P_V \mathbf{R} - P_V P'_{RM_H} \mathbf{R}) \tag{3.30}$$

The above equation is identical to equation (3.18) if $\mathbf{R} = \sigma^2 \mathbf{I}$.

The first term in the above expression is independent of the scanning filters. Since the larger the value of the second term the smaller the value of the mean-square error, the term

$$Trace P_V P'_{RM_H} \mathbf{R}$$

can be used as a measure of goodness of the set of scanning filters with respect to the given data set. Expression (3.30) is the average value of a non-negative quantity and is hence always non-negative. This implies that the maximum value of the above expression is

$$Trace P_V \mathbf{R}$$

The example in section 3.2.2 shows that this bound can be achieved. A normalized measure is

$$\phi(\mathbf{V},\ \mathbf{R},\ \mathbf{M}_H) = \frac{Trace P_V P'_{RM_H} \mathbf{R}}{Trace P_V \mathbf{R}} \tag{3.31}$$

Comparing the above equation to equation (3.22) it is clear that in this case the space analogous to $R(\mathbf{M}_H)$ is $R(\mathbf{R}\mathbf{M}_H)$ and the distance measure in this space is affected by $\mathbf{R}$. In the definition of the measure ϕ, as in the definition of the measure ν, the fundamental is the goal of the reconstruction. This implies that orthogonal directions decided error weighting in the HVISS. Hence the distance measure in the HVISS is not affected in the above expression for the measure ϕ as compared to the expression (3.22) for ν. Comparing (3.31) to (3.22) it can be seen that

$$\nu(\mathbf{V},\ \mathbf{M}_H) \ = \ \phi(\mathbf{V},\ \sigma^2\mathbf{I},\ \mathbf{M}_H)$$

as expected.

If $\{\mathbf{g}_i\}_{i=1}^{\gamma}$ is an orthonormal basis for $R(\mathbf{RM}_H)$ with respect to the inner product in (3.28) (i.e. $\mathbf{G}^T\mathbf{R}^{-1}\mathbf{G} \ = \ \mathbf{I}$ or the $\mathbf{g}_i$ are '$\mathbf{R}^{-1}$ orthonormal', and $R(\mathbf{G}) \ = \ R(\mathbf{RM}_H)$), then, from Appendix, Theorem 6,

$$P'_{RM_H} \ = \ \mathbf{G}(\mathbf{G}^T\mathbf{R}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{R}^{-1} \ = \ \mathbf{G}\mathbf{G}^T\mathbf{R}^{-1} \tag{3.32}$$

Hence,

$$Trace P_V P'_{RM_H}\mathbf{R} \ = \ Trace \mathbf{N}\mathbf{N}^T\mathbf{G}\mathbf{G}^T$$

If $\eta_i \ = \ ||\mathbf{g}_i||^2$ and $q(\mathbf{g}_i)$ is the q-factor [22] of $\mathbf{g}_i$, equation (3.31) implies that

$$\phi(\mathbf{V},\ \mathbf{R},\ \mathbf{M}_H) \ = \ \frac{\sum_{i=1}^{\gamma}\eta_i q(\mathbf{g}_i)}{Trace P_V\mathbf{R}}$$

from Theorem 6. Comparing the above with equation (3.25), it is clear that the differences due to correlation are incorporated in the directions of the vectors $\mathbf{g}_i$ which are $\mathbf{R}^{-1}$ orthogonal, and in weighting factors η_i, which come about because the $\mathbf{g}_i$ are not normal with respect to the euclidean norm. Further, the vectors $\mathbf{g}_i$ span the space $R(\mathbf{RM}_H)$ unlike the vectors $\mathbf{o}_i$ which span $R(\mathbf{M}_H)$.

The matrix $\mathbf{R}$ is positive semi-definite. This implies that it has a real 'square root', i.e. a real matrix $\mathbf{X}$ exists such that $\mathbf{X}^2 \ = \ \mathbf{R}$. Denote $\mathbf{X}$ as $\sqrt{\mathbf{R}}$. It may be noted that the columns of $\sqrt{\mathbf{R}}$ form an $\mathbf{R}^{-1}$ orthonormal basis for N-space, i.e.

$R(\sqrt{\mathbf{R}}) = R(\mathbf{I})$ and $\sqrt{\mathbf{R}}\mathbf{R}^{-1}\sqrt{\mathbf{R}} = \mathbf{I}$. The above expression may be rewritten as

$$\phi(\mathbf{V},\ \mathbf{R},\ \mathbf{M}_H) = \frac{\sum_{i=1}^{\gamma}\sum_{j=1}^{\alpha}\mathbf{t}_{ij}^2}{TraceP_V\mathbf{R}} = \frac{\sum_{i=1}^{\gamma}\sum_{j=1}^{\alpha}\mathbf{t}_{ij}^2}{\sum_{i=1}^{N}\sum_{j=1}^{\alpha}\mathbf{u}_{ij}^2}$$

where $\mathbf{t}_{ij}$ are the α-stimulus values of the vectors $\mathbf{g}_i$ and $\mathbf{u}_{ij}$ are the α-stimulus values of the columns of $\sqrt{\mathbf{R}}$, with respect to orthogonal directions $(\mathbf{n}_j)$ in $R(\mathbf{V})$. Compare the above equation with equation (3.26).

It can be seen that if $\mathbf{R} = \sigma^2\mathbf{I}$, then $\mathbf{G} = \sigma\mathbf{O}$ so that

$$\phi(\mathbf{V},\ \sigma^2\mathbf{I},\ \mathbf{M}_H) = \frac{\sum_{i=1}^{\beta}\sigma^2 q(\mathbf{o}_i)}{\sigma^2 TraceP_V} = \frac{\sum_{i=1}^{\beta}q(\mathbf{o}_i)}{traceP_V}$$

as expected. In the next section, orthogonal directions in the HVISS are not considered, instead the tristimulus values are the goal of the reconstruction and the directions of the CIE matching functions weighted by the viewing illuminant are given importance over arbitrary orthogonal directions in the HVISS.

3.5.2 Data-Dependent Measure Based on Mean-square Tristimulus Error

The tristimulus error is probably a better indicator of the perceptual error ΔE_{Lab} than the fundamental error is. The error between fundamentals weights every direction in the HVISS equally, because orthogonal directions in the HVISS form the basis of the error measure. Averaging over a data set adds weighting factors due to data correlation. The tristimulus error will not weight each direction equally even in the absence of data correlation, because it is based on the directions of the CIE matching functions which are not orthogonal with respect to the regular inner product.

From equations (3.1) and (3.8) the tristimulus error may be written as

$$\mathbf{t} - \hat{\mathbf{t}}_1 = \mathbf{A}_L^T(\mathbf{I} - \mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^{-1}\mathbf{M}_H^T)\mathbf{f}$$

It can be shown that (see Appendix, Theorem 1)

$$E[||\mathbf{t} - \hat{\mathbf{t}}_1||^2] = Trace(\mathbf{A}_L^T\mathbf{R}\mathbf{A}_L - \mathbf{A}_L^T\mathbf{R}\mathbf{M}_H(\mathbf{M}_H^T\mathbf{R}\mathbf{M}_H)^{-1}\mathbf{M}_H^T\mathbf{R}\mathbf{A}_L)$$

Comparing this to equation (3.13) and noticing that equation (3.30) may be rewritten as

$$E[||\mathbf{e}||^2] = Trace(\mathbf{N}\mathbf{N}^T\mathbf{R} - \mathbf{N}\mathbf{N}^T P'_{RM_H}\mathbf{R})$$

it can be seen that relacing $\mathbf{N}$ with $\mathbf{A}_L$ in the above equation gives

$$E[||\mathbf{e}||^2] = Trace(\mathbf{A}_L\mathbf{A}_L^T\mathbf{R} - \mathbf{A}_L\mathbf{A}_L^T P'_{RM_H}\mathbf{R})$$

The above equation is, from equation (3.32),

$$E[||\mathbf{e}||^2] = Trace(\mathbf{A}_L^T\mathbf{A}_L\mathbf{R} - \mathbf{A}_L^T\mathbf{A}_L\mathbf{G}\mathbf{G}^T)$$

Using the reasoning of section 3.5.1 it is clear that $Trace(\mathbf{A}_L^T\mathbf{A}_L\mathbf{G}\mathbf{G}^T)$ is a measure of the goodness of a filter set with respect to the accuracy of the estimated tristimulus values. The maximum value of the expression is $Trace(\mathbf{A}_L\mathbf{A}_L^T\mathbf{R})$. Hence a measure of the goodness of a filter set with respect to mean-square error in CIE tristimulus values is

$$\tau(\mathbf{A}_L, \mathbf{M}_H, \mathbf{R}) = \frac{Trace(\mathbf{A}_L\mathbf{A}_L^T\mathbf{G}\mathbf{G}^T)}{Trace(\mathbf{A}_L\mathbf{A}_L^T\mathbf{R})} = \frac{Trace(\mathbf{A}_L\mathbf{A}_L^T P'_{RM_H}\mathbf{R})}{Trace(\mathbf{A}_L\mathbf{A}_L^T\mathbf{R})} \tag{3.33}$$

which is also

$$\tau(\mathbf{A}_L, \mathbf{M}_H, \mathbf{R}) = \frac{\sum_{i=1}^{\gamma}\sum_{j=1}^{\alpha}\mathbf{t}_{ij}^2}{\sum_{i=1}^{N}\sum_{j=1}^{\alpha}\mathbf{u}_{ij}^2}$$

where the $\mathbf{t}_{ij}$ are CIE tristimulus values of the vectors $\mathbf{g}_i$ (which form an $\mathbf{R}^{-1}$ orthonormal basis for $R(\mathbf{RM})$), and the $\mathbf{u}_{ij}$ are CIE tristimulus values of the columns of $\sqrt{\mathbf{R}}$ which form an $\mathbf{R}^{-1}$ orthonormal basis for N-space.

3.5.3 Generalization of Measures Based on Mean-square Errors

In conclusion one may note that measures based on mean-square errors of colour reproductions are similar in many respects. The directions in the HVISS (orthogonal directions for the fundamental and CIE tristimulus directions for the CIE tristimulus values) determine the inner-products in the HVISS and data correlation decides orthonormality in the 'scanning space'. All such normalized measures may be written in the form of equation (3.26):

$$\frac{\sum_{i=1}^{\gamma}\sum_{j=1}^{s}\mathbf{t}_{ij}^2}{\sum_{i=1}^{N}\sum_{j=1}^{s}\mathbf{u}_{ij}^2}$$

where the $\mathbf{t}_{ij}$ are the s-stimulus values of the vectors $\mathbf{g}_i$ (an $\mathbf{R}^{-1}$ orthonormal basis for $R(\mathbf{RM}_H)$), and the $\mathbf{u}_{ij}$ are s-stimulus values of the columns of $\sqrt{\mathbf{R}}$ (an $\mathbf{R}^{-1}$ orthonormal basis for N-space).

Recall that α is the dimension of $R(\mathbf{V})$. In the specific case when the fundamental is the goal of the reproduction, $s = \alpha$ in the above equation, and the s-stimulus values are the α-stimulus values with respect to orthogonal directions in $R(\mathbf{V})$. In the specific case where the fundamental is the goal of the reconstruction and the statistics of the data set are not known *a priori*, the above expression reduces to the data-independent measure introduced in section 3.2.

3.5.4 Average ΔE_{Lab} Error

It is worthwhile to investigate the possibility of a measure based on a perceptual error. In this section the possibility of a measure based on the average square ΔE_{Lab} error over a given data set is considered. It is shown that it is not feasible to use such a measure as an optimization criterion, because it would not provide a tractable optimization criterion.

Let $\mathcal{F}$ be the transformation from tristimulus space to CIE $L^*a^*b^*$ space (see section 1.1.2) with respect to white point $\mathbf{t}_n = [x_n, y_n, z_n]^T$. The point in $L^*a^*b^*$ space corresponding to tristimulus vector $\mathbf{x}$ is given by equation (1.4). The error vector in CIE $L^*a^*b^*$ space due to the tristimulus estimate of equation (3.8) is

$$\mathcal{F}(\mathbf{t}) - \mathcal{F}(\hat{\mathbf{t}}_1) = \mathbf{\Upsilon}((\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}} - (\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{1}{3}}) \tag{3.34}$$

for the linear transformation $\mathbf{\Upsilon}$ of equation (1.5). Recall that $(\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}}$ implies term-by-term division and term-by-term cube roots.

The average square ΔE_{Lab} error is the average value of the square of the 2-norm of the above vector over the given data set:

$$\frac{1}{n}\sum_{\mathbf{f}} \Delta E_{Lab}^2 = \frac{1}{n}\sum_{\mathbf{f}} Trace\mathbf{\Upsilon}((\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}} - (\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{1}{3}})((\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}} - (\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{1}{3}})^T\mathbf{\Upsilon}^T$$

where n is the number of data points in the set. The above expression may be rewritten as

$$\frac{1}{n}\sum_{\mathbf{f}} \Delta E_{Lab}^2 = \frac{1}{n}\sum_{\mathbf{f}} Trace\mathbf{\Upsilon}(\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}}((\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}})^T\mathbf{\Upsilon}^T - 2Trace\mathbf{\Upsilon}(\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}}((\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{1}{3}})^T\mathbf{\Upsilon}^T$$

$$+ Trace\mathbf{\Upsilon}(\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{1}{3}}((\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{1}{3}})^T\mathbf{\Upsilon}^T$$

There is no simple formula for the cube root of the linear estimate $\hat{\mathbf{t}}_1$. Hence calculating the above error requires a numerical process which is not much simpler than finding the ΔE_{Lab} error for a given data set. A measure based on the above formulation of the ΔE_{lab} error is not amenable to simplification and will not provide a formula that is any simpler than the direct calculation of the ΔE_{Lab} values and averaging them. This implies that a measure based on the ΔE error would not be easy to use in optimization problems and hence in filter design. Besides the reduction in computational complexity involved, simpler formulae often provide an insight into the system behaviour.

3.5.5 Experimental Results

The ensembles used in section 3.4 were used to study the appropriateness of the proposed data-dependent measures, $\phi(\mathbf{A}_L, \mathbf{R}, \mathbf{M}_H)$ and $\tau(\mathbf{A}_L, \mathbf{R}, \mathbf{M}_H)$ and to compare them to the data-independent measure $\nu(\mathbf{A}_L, \mathbf{M}_H)$. The filter sets shown here are the same filter sets used in Section 4. Table 3.7 lists the values of the normalizing factors ($TraceP_V\mathbf{R}$ for the measure ϕ and $Trace\mathbf{A}\mathbf{A}^T\mathbf{R}$ for the measure τ) for the various data sets. It also lists the values of $Trace\mathbf{R}$, which is an upper bound for $TraceP_V\mathbf{R}$ (see section 3.2.2). The scanning filter data was corrected to best fit the actual tristimulus values, assuming knowledge of data correlation. The mean-square error e, the average and maximum ΔE_{Lab} errors E and E_{max}, the mean-square tristimulus error t, and the measures $\nu(\mathbf{A}, \mathbf{M}_H)$, $\phi(\mathbf{A}_L, \mathbf{R}, \mathbf{M}_H)$ and $\tau(\mathbf{A}_L, \mathbf{R}, \mathbf{M}_H)$ are tabulated in Tables 3.8-3.12 for the five sets of signals used. Note that the values of e, E and E_{max} are the same as the values of e_2, E_2 and E_{2max} in Tables 3.2-3.6.

The measures ϕ and τ suggested do not discriminate enough to indicate the differences in perceptual error measures among the filter sets with the fairly high measures presented here. In the subsequent discussion explaining the reason for this, the ratio of the actual tristimulus error to the maximum possible tristimulus error, $Trace\mathbf{A}_L\mathbf{A}_L^T\mathbf{R}$, is called the fractional tristimulus error. The term $\frac{\mathbf{t}}{\mathbf{t}_n}$ is called the vector of the 'normalized tristimulus values'. The 'normalized tristimulus error vector'

Table 3.7: Normalizing Factors

Data Set	Trace $\mathbf{R}$	$Trace\ P_V\mathbf{R}$	$Trace\mathbf{AA}^T\mathbf{R}$
Colour Copier	1.81	1.05	14.18
Lithographic Printer	2.51	1.49	20.31
Thermal Printer	1.82	1.09	14.57
Inkjet Printer	3.96	2.12	30.13
Munsell Chips	5.09	3.21	45.03

Table 3.8: Colour Copier Data Set. White Point = [7.84, 7.85, 7.79]

Filter Set	Measure ν	Measure ϕ	Measure τ	e	t	E	E_{max}
1	0.771	0.99293	0.99671	0.00741	0.04668	6.56	27.02
2	0.818	0.99617	0.99811	0.00402	0.02684	4.67	20.56
3	0.858	0.99884	0.99934	0.00121	0.00940	2.94	13.48
4	0.913	0.99995	0.99998	0.00005	0.00032	0.49	2.00
5	0.943	0.99998	0.99998	0.00002	0.00022	0.41	1.83

Table 3.9: Lithographic Printer Data Set. White Point = [7.32, 7.38, 6.99]

Filter Set	Measure ν	Measure ϕ	Measure τ	e	t	E	E_{max}
1	0.771	0.99755	0.99883	0.00365	0.02368	3.94	20.97
2	0.818	0.99853	0.99926	0.00220	0.01501	3.03	17.08
3	0.858	0.99916	0.99957	0.00125	0.00883	2.56	12.60
4	0.913	0.99996	0.99999	0.00007	0.00029	0.54	2.33
5	0.943	0.99999	0.99999	0.00001	0.00014	0.31	1.53

Table 3.10: Thermal Printer Data Set. White Point = [8.99, 8.98, 9.13]

Filter Set	Measure ν	Measure ϕ	Measure τ	e	t	E	E_{max}
1	0.771	0.99101	0.99580	0.00981	0.06120	7.49	59.43
2	0.818	0.99351	0.99686	0.00709	0.04571	6.35	58.63
3	0.858	0.99921	0.99963	0.00086	0.00534	2.65	12.70
4	0.913	0.99940	0.99979	0.00066	0.00310	1.81	7.13
5	0.943	0.99993	0.99993	0.00007	0.00098	0.87	5.50

Table 3.11: Inkjet Printer Data Set. White Point = [8.18, 8.23, 7.93]

Filter Set	Measure ν	Measure ϕ	Measure τ	e	t	E	E_{max}
1	0.771	0.98760	0.99476	0.0263	0.1580	9.29	36.13
2	0.818	0.99025	0.99582	0.0207	0.1259	8.45	31.96
3	0.858	0.99944	0.99971	0.0012	0.00873	2.30	7.26
4	0.913	0.99978	0.99992	0.0005	0.00175	1.21	4.30
5	0.943	0.99992	0.99997	0.0002	0.00087	0.70	2.04

Table 3.12: Munsell Chip Set. White Point = [10.2025, 10.1897, 10.0349]

Filter Set	Measure ν	Measure ϕ	Measure τ	e	t	E	E_{max}
1	0.771	0.99500	0.99770	0.0162	0.1046	5.12	18.64
2	0.818	0.99620	0.99820	0.0121	0.0805	4.32	15.11
3	0.858	0.99930	0.99968	0.0022	0.0143	2.10	8.63
4	0.913	0.99966	0.99990	0.0011	0.00454	1.03	8.80
5	0.943	0.99994	0.99997	0.0002	0.00152	0.49	2.07

is the vector $\frac{\mathbf{t}}{\mathbf{t}_n} - \frac{\mathbf{t}_1}{\mathbf{t}_n}$. A small mismatch between the spaces may lead to small fractional tristimulus errors, which would result in a fairly high value of a data-dependent measure. Small fractional tristimulus errors may, however, give high average ΔE_{Lab} values because a particular tristimulus error in the estimation of a point with low normalized tristimulus values results in a larger ΔE_{Lab} error than the same tristimulus error at a 'brighter' point. This discrepancy between performance evaluation of filter sets based on perceptual error and on data-dependent measures is not as acute for the data-independent measure. This is because data correlation serves to de-emphasize the mismatch between spaces in the case of data-dependent measures.

For a quantitative relationship that explains the above claims, consider the following. From equation (3.34) and the algebraic identity $a - b = (a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{2}{3}})$ it is clear that the error vector in CIE $L^*a^*b^*$ space is

$$\mathcal{F}(\mathbf{t}) - \mathcal{F}(\hat{\mathbf{t}}_1) = \Upsilon \frac{((\frac{\mathbf{t}}{\mathbf{t}_n}) - (\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n}))}{((\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{2}{3}} + (\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{1}{3}}(\frac{\mathbf{t}}{\mathbf{t}_n})^{\frac{1}{3}} + (\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n})^{\frac{2}{3}})}$$

where multiplication and division are point by point. The 2-norm of the above error vector depends not only on the normalized tristimulus error vector, but also on the normalized tristimulus values at which the error occurs. While the minimum mean-square tristimulus error estimate treats a particular value of the 2-norm of $((\frac{\mathbf{t}}{\mathbf{t}_n}) - (\frac{\hat{\mathbf{t}}_1}{\mathbf{t}_n}))$ as the same, independent of where the tristimulus error occurs, this value is weighted in the expression for ΔE_{lab} as above. In general, the same normalized tristimulus error vector will result in a larger ΔE_{Lab} value if it occurs at points with low actual and estimated tristimulus values, and hence points with low tristimulus norm. Hence, one can say that the same value of tristimulus error will not result in a similar ΔE_{Lab} error, unless the norms of the tristimulus vectors are similar. This is indicated by the following examples.

The white point for the Munsell Chip set is the point $[10.2025,\ 10.1897,\ 10.0349]^T$. With Filter Set 1, the corrected tristimulus estimate of the white point is $[10.1619,\ 10.1506,\ 10.0160]^T$, resulting in a tristimulus error vector $[0.0406, 0.0391, 0.0189]^T$ and a square tristimulus error of about 0.0035. This square tristimulus error results in a ΔE_{lab} error of 0.199. Notice that the normalized tristimulus vector of the white point is $[1.0,\ 1.0,\ 1.0]^T$. Consider another point in tristimulus space, also part of the Munsell Chip Set, $[2.2695,\ 1.3852,\ 1.2869]^T$. The normalized tristimulus vector for this point is $[0.2224,\ 0.1359,\ 0.1282]^T$. The estimate of this point is $[2.2707,\ 1.4167,\ 1.2786]^T$, resulting in a tristimulus error vector $[-0.0012, -0.0315, 0.0083]^T$ and a square tristimulus error of about 0.0011. This corresponds to a ΔE_{lab} error of 2.18, which is much larger than the corresponding error at the white point. This is partly due to the fact that the white point is a much brighter point as indicated by the normalized tristimulus vectors. Hence the measure values that indicate a perceptible colorimetric error will be decided at least in part by the overall 'brightness' distribution.

Points of identical tristimulus norm may also give different ΔE_{Lab} errors because errors in one tristimulus value are weighted differently from errors in another tristimulus value. Consider an example. Suppose two of the three tristimulus values of the white point are computed exactly. Suppose the third normalized value is 0.97. The value of $0.97^{1/3}$ is approximately 0.99. If the error lies in computing the first tristimulus value, the error vector in CIE $L^*a^*b^*$ space is $[0, 5, 0]^T$. If the error lies in computing the second tristimulus value, the corresponding error vector in CIE $L^*a^*b^*$ space is $[1.16, -5, 2]^T$. Similarly, the error due to an error in the third tristimulus value is $[0, 0, -2]^T$. The corresponding ΔE_{Lab} errors are 5, 5.5 and 2 respectively. This demonstrates the fact that an error vector of fixed norm at a particular point in tristimulus space can give ΔE_{Lab} errors as varied as 2 and 5.5. An extension of the results presented in this section to other data sets is relevant only if the data sets have a similar distribution of tristimulus values.

3.6 Conclusions

The qualities of a good measure were stated and a data-independent measure was developed which satisfies of these requirements. The proposed measure is a valid measure of goodness of a filter set with respect to the mean-square error between the fundamental and its estimate when the signal components are independent, identically distributed (iid). If the signal components are not iid, the proposed measure is at least as good as the q-factor and does eliminate some of the disadvantages of the q-factor. The proposed measure is an excellent indicator of filter performance when the filter measurements are 'corrected' for a specific data set.

The philosophy behind the definition of the data-independent measure is extended to include data-dependence. The fundamental error measure weights orthogonal directions in the HVISS. The mean-square tristimulus error weights the CIE tristimulus directions. Motivated by this and the fact that different displays will require different tristimulus values (which are defined by the primaries for the display), the data-dependent measure is further extended to include the arbitrary weighting of directions in the HVISS. Simulations illustrate that the data-dependent measures may not distinguish well enough among filters producing very low tristimulus errors. It is demonstrated how, on occasion, low tristimulus errors can produce perceptible errors.